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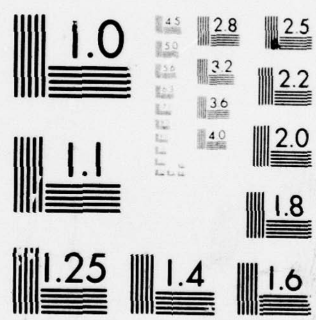

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(6) ON Q-MATRICES, CENTROIDS AND SIMPLOTOPES.

by

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ON Q-MATRICES, CENTROIDS AND SIMPLOTOPES

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1. Background. Our concern in this paper is the question: What is the class, Q, of all real $n \times n$ matrices M such that for every n-vector q the linear complementarity problem (LCP)

$$Iw - Mz = q \tag{1}$$

$$w \geq 0 , \; z \geq 0 \tag{2}$$

$$\langle w, z \rangle = 0 \tag{3}$$

has a solution?

Several researchers have already attacked the problem of characterizing such matrices, and - in a sense - solved it. See [1], [6], [7], [9]. Many more studies have identified subclasses of Q as well as methods for solving the linear complementarity problems formed thereby. But the evidence to date suggests that a "useful" characterization of Q will not be attained by accumulating larger and larger classes of matrices contained in it.

Lately, a frontal assault on the geometry of the problem has begun. In the vanguard of this attack are L. M. Kelly and L. T. Watson to whose papers [5], [6] this research owes much. In particular, we adopt their reformulation of the problem as one of covering the sphere by spherical simplices. We now endeavor to make this statement more precise.

The LCP can be cast in what looks like (but ultimately need not be) more general terms: Given real $n \times n$ matrices A and B and a real n-vector c, find n-vectors w and z such that

$$Aw + Bz = c \tag{1'}$$

$$w \geq 0 , \; z \geq 0 \tag{2'}$$

$$\langle w, z \rangle = 0 \tag{3'}$$

Let us designate the system (1′) - (3′) by the triple (A, B, c) . This means (1) - (3) is the triple $(I, -M, q)$. The question about Q-matrices now becomes: What are necessary and sufficient conditions on A and B to ensure that (A, B, c) has at least one solution for every c ?

Since (2′) and (3′) imply

$$w_i z_i = 0 \qquad i = 1, \ldots, n , \tag{4}$$

attention turns to what are loosely called <u>complementary submatrices</u> of $[A, B]$ and the cones they span. An $n \times n$ matrix C is said to be a <u>complementary submatrix</u> of $[A, B]$ if and only if

$$C_i \in \{A_i, B_i\} . \tag{5}$$

In (5), as elsewhere, A_i , B_i and C_i denote the i^{th} columns of A , B and C , respectively. In principle, there are 2^n complementary submatrices of $[A, B]$; but, if $A_i = B_i$ for some i , there will be fewer than 2^n distinct complementary submatrices of $[A, B]$. At the moment, such duplications are of no consequence. We let

$$\text{comp } [A, B] \tag{6}$$

denote all $n \times n$ matrices C formed in accordance with (5).

Associated with each $C \in \text{comp } [A, B]$ is the so-called <u>complementary cone</u>

$$\text{pos } C = \{c: c = Cx , x \geq 0\} . \tag{7}$$

Solving (A, B, c) can be regarded as a matter of finding $C \in \text{comp}\,[A, B]$ such that $c \in \text{pos}\, C$. In fact, the union of the complementary cones (relative to $[A, B]$) is precisely the set of n-vectors c for which (A, B, c) has a solution. What interests us is finding necessary and sufficient conditions on A and B for which

$$\cup\,\{\text{pos}\, C\colon C \in \text{comp}\,[A, B]\} = E_n\,, \tag{8}$$

where E_n denotes Euclidean n-space. This formulation of the problem was first used by Samelson, Thrall and Wesler [10] and later by Murty [8].

A nonsingular matrix $C \in \text{comp}\,[A, B]$ is called a complementary basis in $[A, B]$. Notice that if A is nonsingular, the problem (A, B, c) can be transformed to $(I, A^{-1}B, A^{-1}c)$ which more closely resembles $(I, -M, q)$.

Normally, the matrix M in $(I, -M, q)$ is called nondegenerate if and only if every principal minor of M is nonzero. It is easy to see that M is nondegenerate if and only if every member of $\text{comp}\,[I, -M]$ is nonsingular. With this in mind, we call $[A, B]$ nondegenerate if and only if

$$\det C \neq 0 \quad \text{for every} \quad C \in \text{comp}\,[A, B]\,. \tag{9}$$

Otherwise $[A, B]$ is degenerate.

From the formulation of (A, B, c) it is evident that the columns A_i, B_i $(i = 1, \ldots, n)$ and c can be positively scaled without affecting the solvability of the problem. Now, since (A, B, c) is trivial when $c = 0$, we may assume

$$\|c\| = 1 \ . \tag{10}$$

Furthermore, it can be shown [12] that (8) cannot hold if any column of [A, B] is 0 . Hence in the present investigation, we shall assume

$$\|A_i\| = \|B_i\| = 1 \qquad i = 1, \ldots, n \ . \tag{11}$$

This means that c and all the columns of [A, B] can be regarded as points on the unit (n-1)-sphere

$$S^{n-1} = \{x \in E_n : \|x\| = 1\}$$

with center at the origin, 0 .

<u>For the present, we strengthen the assumption</u> (11) <u>by assuming that [A, B] is nondegenerate</u>. Although this is a strong assumption, it is far from being powerful enough to imply much vis-à-vis the existence of solutions. Under the present nondegeneracy assumption, if $C = [C_1, \ldots, C_n] \in \text{comp}\ [A, B]$, then the vectors $C_1, \ldots, C_n$ are linearly independent and (by virtue of their uniform length) can be considered as the vertices of a <u>spherical</u> (n-1)<u>-simplex</u>, $\sigma(C)$. Note that

$$\sigma(C) = \text{pos}\ C \cap S^{n-1} \ . \tag{12}$$

In line with (8), the question can be put as follows: What are necessary and sufficient conditions on A and B to ensure that

$$\cup \{\sigma(C) : C \in \text{comp}\ [A, B]\} = S^{n-1} \ ? \tag{13}$$

We want to know when the collection of spherical (n-1)-simplices $\sigma(C)$ covers the (n-1)-sphere S^{n-1}.

Following Kelly and Watson [6] to whom this interpretation of the problem is due, we shall call comp [A, B] a Q-arrangement if and only if (13) holds; and, having taken the liberty of using the term "non-degenerate" in reference to [A, B], we go a step further and call [A, B] a Q-matrix if comp [A, B] is a Q-arrangement. It is clear that with A being nonsingular, [A, B] is a Q-matrix in the new sense if and only if $-A^{-1}B$ is a Q-matrix in the original sense.

We propose to investigate the characterization of Q-matrices according to the formulation given by (13). Indeed, we state a geometrical conjecture on necessary and sufficient conditions for (13) to hold. We show that the condition is only necessary in general and that it is also sufficient in the case $n = 2$.

In order to state the conjecture, we continue with the development of the geometrical structures involved. Let

$$\mathscr{B}^n = \{x \in E_n : \|x\| \leq 1\}$$

be the unit ball in E_n with center at 0. Note that S^{n-1} is the boundary of $\mathscr{B}^n$. Now for each $C \in$ comp [A, B] define

$$\tau(C) = \text{pos } C \cap \mathscr{B}^n . \tag{14}$$

We call such sets spherical sectors. Our nondegeneracy assumption implies that the $\tau(C)$ are n-dimensional, and of course they are compact and convex. Thus they are convex bodies [2]. Moreover, their union is $\mathscr{B}^n$

if and only if $[A, B]$ is a Q-matrix.

Now we regard $\mathscr{B}^n$ as a convex body with homogeneous distribution of mass. For each $C \in \text{comp}\,[A, B]$ let

$$\bar{x}(C) = \text{centroid (center of mass) of } \tau(C) . \tag{15}$$

If $v(C)$ denotes the volume of $\tau(C)$, then

$$\bar{x}(C) = \frac{1}{v(C)} \int_{\tau(C)} x\,dx . \tag{16}$$

For the moment, we say no more about how these centroids are actually found. Finally, we define

$$X[A, B] = \text{conv}\,\{\bar{x}(C) : C \in \text{comp}\,[A, B]\} . \tag{17}$$

That is, $X[A, B]$ is the convex hull of the centroids of the spherical sectors relative to $[A, B]$. At last we can state the conjecture we want to study.

<u>Conjecture</u>. Let $[A, B]$ be nondegenerate. Then

$$[A, B] \text{ is a Q-matrix if and only if } 0 \in \text{int}\,X[A, B] . \tag{18}$$

As mentioned earlier, we shall prove that (18) holds when $n = 2$ and also prove that for all $n \geq 2$ the necessity $(0 \in \text{int}\,X[A, B])$ must hold. We shall also investigate the converse and prove it is false for $n \geq 3$.

2. Proof of the conjecture for the case $n = 2$. It must be conceded that a criterion for Q-matrices is not very interesting in the case where $n = 2$. There are several reasons for this. Among them are the following arguments:

1° One can draw the complementary cones pos C and see whether their union is E_2.

2° There already exist finite numerical tests for whether a 2×2 matrix M belongs to Q. (See [1], [6].) The transformation indicated in Section 1 coupled with such a test would handle the "more general" case of [A, B].

To these may be added another objection:

3° The result - being a theorem in plane geometry - is probably already known. (The authors have not yet found a reference, however.)

These points notwithstanding, there is a simple rationale for establishing (18) for $n = 2$: It makes the ideas more concrete.

In the 2-dimensional case, the "spherical" 1-simplices $\sigma(C)$ are just closed arcs on the unit circle. The nondegeneracy assumption means that for each $C = [C_1, C_2] \in \text{comp}\,[A, B]$, C_1 and C_2 are neither identical nor antipodal points. The length of $\sigma(C)$ must be positive and less than π. A "spherical" sector $\tau(C)$ is just an ordinary circular sector, and its centroid lies on the line segment between the center of the circle and the midpoint of $\sigma(C)$.

Theorem 1. If A and B are real 2×2 matrices such that [A, B] is nondegenerate then

$$[A, B] \text{ is a Q-matrix if and only if } 0 \in \text{int}\, X[A, B].$$

Proof. Suppose $[A, B]$ is a Q-matrix. If $0 \notin \text{int } X[A, B]$, then $X[A, B]$ must lie within a half-disk, the region bounded by a semi-circle and the corresponding diameter, $\overline{ST}$. Let N denote the midpoint of the other semi-circle. Thus $\overline{ON} \perp \overline{ST}$ and $\overline{ST}$ strictly separates N from $X[A, B]$. Now since $[A, B]$ is assumed to be a Q-matrix, there exists a matrix $C \in \text{comp } [A, B]$ such that $N \in \sigma(C)$. Let R be the midpoint of $\sigma(C)$. Then $\bar{x}(C)$, the centroid of $\sigma(C)$, lies on the radius $\overline{OR}$, and as $\bar{x}(C)$ belongs to $X[A, B]$, $\overline{ST}$ strictly separates $\bar{x}(C)$ and thus R from N . But this implies the length of $\sigma(C)$ is greater than π which is impossible. Thus $0 \in \text{int } X[A, B]$. (See Figure 1.)

Conversely, suppose $0 \in \text{int } X[A, B]$ and $[A, B]$ is not a Q-matrix. Then there exists a point N on S^1 which does not belong to any of the arcs $\sigma(C)$. Since $0 \in \text{int } X[A, B]$, there cannot be an entire semi-circle of such "uncovered" points. Hence any uncovered point such as N must belong to an open minor arc whose endpoints are A_i and B_i ; say $i = 1$. The antipodal points corresponding to A_1 and B_1 are $-A_1$ and $-B_1$, respectively. Since N is uncovered, it follows that both A_2 and B_2 must belong to the open arc from $-A_1$ to $-B_1$. But this forces all the centroids $\bar{x}(C)$ to lie in the same half-disk, namely the one determined by the bisector $\overline{ST}$ of the central angle $-A_1 0 B_1$ and containing $-A_1$ and $-B_1$. (See Figure 2.) This is impossible if $0 \in \text{int } X[A, B]$. $\square$

Theorem 1 establishes the conjecture (18) for the limited case of $n = 2$. We mention in passing that one can readily see how much the nondegeneracy assumption can be relaxed. When $n = 2$ one needs at least three distinct nonsingular members of comp $[A, B]$.

3. Proof of necessity in the general case. As we shall demonstrate in this section, the separation argument used in the first half of the proof of Theorem 1 lends itself to higher-dimensional generalization. For any $n \geq 2$, suppose $[A, B]$ is a nondegenerate* Q-matrix and $0 \notin \operatorname{int} X[A, B]$. Then by standard separation arguments there exists a hyperplane H which separates 0 from $X[A, B]$, i.e., 0 and $X[A, B]$ do not lie in the same open half-space with respect to H . In fact, we may choose H to satisfy

$$H = \{x: \langle d, x\rangle = 0\} \tag{19}$$

$$\|d\| = 1 \tag{20}$$

$$\langle d, y\rangle \leq 0 \qquad \text{for all} \quad y \in X[A, B] . \tag{21}$$

This means H strictly separates d from $X[A, B]$. Now since $[A, B]$ is supposed to be a Q-matrix, $d \in \sigma(C)$ for some $C \in \operatorname{comp}[A, B]$. Let $\tau(C)$ be the corresponding spherical sector, and let $\bar{x}(C)$ be its centroid. Since $\bar{x}(C) \in X[A, B]$ the inequality

$$\langle d, \bar{x}(C)\rangle > 0 \tag{22}$$

would clearly contradict (21). As we shall show, this is precisely what

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*The full strength of the nondegeneracy assumption is not needed here. What we want to do is restrict attention to the nonsingular $C \in \operatorname{comp}[A, B]$. If $[A, B]$ is a Q-matrix, then it follows that S^{n-1} is covered by the spherical (n-1)-simplices $\sigma(C)$ corresponding to these nonsingular $C \in \operatorname{comp}[A, B]$. In this more natural setting $X[A, B]$ would be the convex hull of the corresponding $\bar{x}(C)$.

is implied by the hypotheses stated above. To this end, we first prove

Lemma 1. The inequality (22) holds for all $d \in \sigma(C)$ if and only if

$$\langle C_i, \bar{x}(C) \rangle > 0 \qquad i = 1, \ldots, n . \tag{23}$$

Proof. The necessity is obvious since the C_i are points of $\sigma(C)$. For the sufficiency, note that for any $d \in \sigma(C)$ we have

$$d = \sum_{i=1}^{n} \lambda_i C_i , \quad \lambda_i \geq 0 , \quad \sum_{i=1}^{n} \lambda_i > 0 . \tag{24}$$

Combining (23) and (24) we obtain (22). □

Thus, our task comes down to establishing (23). In geometrical terms, (23) says that the centroid of a spherical sector makes an acute angle with the vertices $C_1, \ldots, C_n$. (Recall that in the case of $n = 2$, this is obvious since $\bar{x}(C)$ lies on the bisector of a central angle of less than π radians.)

At this point, our work requires only the nonsingularity of $C = [C_1, \ldots, C_n]$, the associated sets $\sigma(C)$, $\tau(C)$, and of course $\bar{x}(C)$. The inequality (23) is a geometrical statement independent of the complementarity setting. It should also be pointed out that the fact we are about to establish may already be known, but the authors have not succeeded in locating it in the literature.

The proof will be made somewhat tidier if we settle a few minor

points beforehand. First, it is clear that a proof of (23) for $i = n$ which makes no special assumptions about C_n can be adapted to prove $\langle C_i, \bar{x}(C)\rangle > 0$ for all i. Second, the proof makes use of an appropriately defined wedge containing $\tau(C)$. The wedge is split into parts, each of which is a convex body in its own right. In this way, we relate the centroids of the parts to the centroid of the entire wedge. This device makes use of

Lemma 2. Let K be a homogeneous convex body with volume $v(K)$ and centroid $\bar{x}(K)$. If K is partitioned into m convex bodies $K_1, \ldots, K_m$ with volumes $v(K_1), \ldots, v(K_m)$ and centroids $\bar{x}(K_1), \ldots, \bar{x}(K_m)$ respectively, then

$$\bar{x}(K) = \sum_{i=1}^{m} \frac{v(K_i)}{v(K)} \bar{x}(K_i) . \tag{25}$$

Proof. This follows from (16). □

Note that since $v(K) = \sum_{i=1}^{m} v(K_i)$, the equation (25) means that $\bar{x}(K)$ is a convex combination of the $\bar{x}(K_i)$, $i = 1, \ldots, m$.

We are now ready to state and prove the result.

Theorem 2. $\langle C_n, \bar{x}(C)\rangle > 0$.

Proof. We begin by defining the (n-1)-dimensional subspace

$$H_0 = \{x \in E_n : \langle C_n, x\rangle = 0\} .$$

The hyperplane H_0 is just the orthogonal complement of the 1-dimensional space generated by C_n . It determines two half-spaces

$$H_0^+ = \{x \in E_n : \langle C_n, x\rangle \geq 0\}$$

and

$$H_0^- = \{x \in E_n : \langle C_n, x\rangle \leq 0\} .$$

There are three cases determined by the way in which the vertices $C_1, \ldots, C_{n-1}$ are situated with respect to these half-spaces. To describe these cases efficiently, we introduce another harmless assumption. Indeed, we may assume without loss of generality that the vertices are labeled in such a way that

$$\langle C_i, C_n\rangle \leq \langle C_{i+1}, C_n\rangle \qquad i = 1, \ldots, n-2 . \tag{26}$$

<u>Case</u> 1. $\langle C_1, C_n\rangle \geq 0$. This case is trivial since $\bar{x}(C)$ is a <u>positive</u> linear combination of the C_i $(1 \leq i \leq n)$. In view of the fact that $\langle C_n, C_n\rangle = 1$, we have the required inequality $\langle C_n, \bar{x}(C)\rangle > 0$.

<u>Case</u> 2. $\langle C_1, C_n\rangle < 0$ and $\langle C_{n-1}, C_n\rangle \leq 0$. Here we define a set of n more distinct (n-1)-dimensional subspaces (hyperplanes). For

$i = 1, \ldots, n$ let

$$H_i = \{x \in E_n : x = \sum_{j=1}^{n} C_j y_j \; , \; y_i = 0\} \; .$$

Each hyperplane can be put in the form:

$$H_i = \{x \in E_n : \langle p_i, x\rangle = 0\} \qquad i = 1, \ldots, n$$

where the vector p_i satisfies

$$\langle p_i, C_i\rangle > 0 \qquad i = 1, \ldots, n \; .$$

Again each of these hyperplanes gives rise to a pair of half-spaces:

$$H_i^+ = \{x \in E_n : \langle p_i, x\rangle \geq 0\} \qquad i = 1, \ldots, n$$

$$H_i^- = \{x \in E_n : \langle p_i, x\rangle \leq 0\} \qquad i = 1, \ldots, n \; .$$

Note that the H_i $(i = 1, \ldots, n)$ are the bounding hyperplanes of pos C . Moreover,

$$\tau : = \tau(C) = \mathscr{B}^n \bigcap_{i=1}^{n} H_i^+ \; .$$

Using the half-spaces $H_1^+, \ldots, H_{n-1}^+$ we define the <u>spherical</u> <u>wedge</u>

$$W = \mathscr{B}^n \bigcap_{i=1}^{n-1} H_i^+ .$$

(We note in passing that $\tau = W \cap H_n^+$.) The wedge W is n-dimensional. Let $\bar{x}_W$ denote its centroid. It is easy to verify the following important facts about W . First, W contains the diameter of $\mathscr{B}^n$ passing through C_n (and the antipodal point $-C_n$). Second, the hyperplane H_0 splits W into two "congruent" parts. Third (and really crucial),

$$\langle C_n, \bar{x}_W \rangle = 0 . \tag{27}$$

The latter follows from Lemma 2 and the remark about congruence.

Now notice that if we let

$$\gamma : = \mathscr{B}^n \bigcap_{i=1}^{n-1} H_i^+ \cap H_n^- = W \cap H_n^-$$

then we can write

$$W = \tau \cup \gamma . \tag{28}$$

The set γ is also a convex body. Let $\bar{x}_\tau = \bar{x}(C)$, and let $\bar{x}_\gamma$ denote the centroid of γ . Then

$$\bar{x}_W = \frac{v(\tau)}{v(W)} \bar{x}_\tau + \frac{v(\gamma)}{v(W)} \bar{x}_\gamma \tag{29}$$

where $v(\tau)$, $v(\gamma)$, and $v(W)$ are the volumes of τ , γ , and W , respectively. Since γ is a subset of H_0^- (by the assumptions of the present case) we have

$$\langle C_n, \bar{x}_\gamma \rangle < 0 . \tag{30}$$

The desired result now follows from (27), (29), and (30). (Figure 3 depicts the situation for the case where $n = 3$.)

Case 3. $\langle C_1, C_n \rangle < 0 < \langle C_{n-1}, C_n \rangle$. This hypothesis means that the open half-spaces relative to H_0 each contain at least one point from the set $\{C_1, \ldots, C_{n-1}\}$. Under these circumstances, the idea is to reduce the proof to the two preceding cases by introducing some new points and corresponding convex bodies. (See Figure 4.) A spherical sector to which Case 1 applies will be called a set of type 1, and a sector to which Case 2 applies will be called a set of type 2.*

Of particular interest is

$$\{P_{ij}: P_{ij} = H_0 \cap \sigma(C_i, C_j) , \ C_i \in \text{int } H_0^- , \ C_j \in \text{int } H_0^+ , \ j \neq n\} \tag{31}$$

*In Figure 4, the sector with vertices $0, P_{12}, C_2, C_3$ is of type 1 whereas that with vertices $0, C_1, P_{12}, C_3$ is of type 2.

where $\sigma(C_i, C_j)$ denotes the (1-dimensional) edge of $\sigma(C)$ with endpoints C_i and C_j. Each P_{ij} defined in (31) is the central projection onto S^{n-1} of a point C_{ij} lying in $H_0 \cap H_n$ on the line segment between C_i and C_j. The points $C_1, \ldots, C_{n-1}$ are vertices of an ordinary (n-2)-simplex $\Delta(C_1, \ldots, C_{n-1})$ lying in H_n. What we want to do is use the points $C_1, \ldots, C_{n-1}$ and the P_{ij} to define a partitioning of $\sigma(C_1, \ldots, C_{n-1})$ into (n-2)-simplices such that all the vertices of each individual (n-2)-simplex lie either in H_0^+ or H_0^-. It will suffice to do this to $\Delta(C_1, \ldots, C_{n-1})$ using only $C_1, \ldots, C_{n-1}$ and the C_{ij} as vertices of the (n-2)-simplices in the partitioning.

We claim that a partitioning of this sort exists. (See the Appendix.) By adjoining 0 and C_n to each of the (n-2)-simplices (of the partition) that lie in H_0^+, we obtain spherical sectors of type 1. The centroid of each such sector belongs to int H_0^+. Similarly, by adjoining 0 and C_n to each of the other (n-2)-simplices (of the partition) we obtain spherical sectors of type 2. (Each of these is the portion of a wedge lying in H_0^+.) By Case 2, the centroid of such a set belongs to int H_0^+. Assembling all the aforementioned spherical sectors (on both sides of H_0) we obtain the given sector τ. Now, by Lemma 2,

$$\bar{x}(C) = \bar{x}_\tau \in \text{int } H_0^+$$

and this completes the proof. □

4. <u>Disproof of the sufficiency in the general case</u>. In the preceding section, we remarked that the nondegeneracy assumption could be dropped provided that $X[A, B]$ were properly redefined. Let us assume this is done.

It is clear that the property $0 \in \text{int } X[A, B]$ is not adversely affected by slight perturbation of some columns of $[A, B]$. This means that if there exists a Q-matrix, $[A, B]$, having an arbitrarily small perturbation $[\tilde{A}, \tilde{B}]$ which is <u>not</u> a Q-matrix, then the property $0 \in \text{int } X[A, B]$ cannot be sufficient for $[A, B]$ to be a Q-matrix.

The existence of just such a matrix has been discovered by Watson [13]. Let $[A, B] = [I, -M]$ where

$$M = \begin{pmatrix} 1 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

The matrix M is degenerate since two of its principal minors are 0. To see that M is a Q-matrix (equivalently that comp $[I, -M]$ is a Q-arrangement) takes a bit of work which we shall omit. The important point though is that for $\varepsilon > 0$, the matrix

$$M(\varepsilon) = \begin{pmatrix} 1 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & \varepsilon & 0 \end{pmatrix}$$

is not in Q since $(I, -M(\varepsilon), q(\varepsilon))$ has no solution for

$$q(\varepsilon) = \begin{pmatrix} 0 \\ 1 \\ -\varepsilon/4 \end{pmatrix}, \quad \varepsilon > 0 .$$

This example shows that for $n \geq 3$, $0 \in \text{int } X[A, B]$ does <u>not</u> imply that $[A, B]$ is a Q-matrix. In a sense this is easy to "explain." In this example, there are six centroids $\bar{x}(C)$ where $C \in \text{comp } [I, -M]$ and $\det C \neq 0$. The origin must belong to the convex hull of four centroids (though not necessarily to its interior). At any rate, it seems as though there could be enough degrees of freedom to insure that $0 \in \text{int } X[A, B]$ and still arrange the vectors A_i, B_i, $i = 1, 2, 3$ in such a way that $[A, B]$ is not a Q-matrix.

<u>Appendix</u>: <u>Existence of the partitioning</u>. Here we wish to prove that the partitioning referred to in the proof of Theorem 2 (Case 3) actually exists. As we shall show, the matter before us has strong connections to very classical constructions such as the partitioning of a quadrilateral into two triangles (Figure 5) and the partitioning of a triangular prism into three tetrahedra (Figure 6).

In order to clarify the relevance of what we prove in this Appendix, we review the hypotheses of Case 3. We have an (ordinary) (n-1)-simplex $\Delta(C_1, \ldots, C_n)$ and a hyperplane

$$H_0 = \{x \in E_n : \langle C_n, x \rangle = 0\} .$$

The vertices of the simplex are labeled so that

$$\langle C_n, C_i \rangle \leq \langle C_n, C_{i+1} \rangle \qquad i = 1, \ldots, n-2 .$$

The hypothesis that defines Case 3 is

$$\langle C_n, C_1 \rangle < 0 < \langle C_n, C_{n-1} \rangle .$$

Thus

$$C_1 \in \operatorname{int} H_0^- \quad \text{and} \quad C_{n-1} \in \operatorname{int} H_0^+ .$$

We may partition the vertices $C_1, \ldots, C_{n-1}$ as follows:

$$\begin{aligned} V^- &= \{C_i : \langle C_n, C_i \rangle < 0\} = \{C_1, \ldots, C_k\} \\ V^0 &= \{C_i : \langle C_n, C_i \rangle = 0\} = \{C_{k+1}, \ldots, C_\ell\} \\ V^+ &= \{C_i : \langle C_n, C_i \rangle > 0\} = \{C_{\ell+1}, \ldots, C_{n-1}\} \end{aligned}$$

with the stipulations $1 \leq k < \ell+1 \leq n-1$ and the understanding that V^0 may be empty in which case $k = \ell$.

Setting C_n aside temporarily, we concentrate on the face $\Delta(C_1, \ldots, C_{n-1})$ which is "split" into two pieces by H_0 . That is, $\Delta(C_1, \ldots, C_{n-1})$ has vertices lying in each of the open half-spaces determined by H_0 . Therefore, if $(C_i, C_j) \in V^- \times V^+$, the edge $\Delta(C_i, C_j)$ meets H_0 in a point which we denote by C_{ij} . We now define the set

$$G := \{C_{ij} \in E^n : C_{ij} = H_0 \cap \Delta(C_i, C_j) \ , \ \ (C_i, C_j) \in V^- \times V^+\} .$$

The cardinality of G is $k(n-1-\ell)$.

Our aim is to partition

$$\Delta^+(C_1, \ldots, C_{n-1}) := H_0^+ \cap \Delta(C_1, \ldots, C_{n-1})$$

and

$$\Delta^-(C_1, \ldots, C_{n-1}) := H_0^- \cap \Delta(C_1, \ldots, C_{n-1})$$

<u>into</u> (n-2)<u>-simplices whose vertices all belong to</u> $V^+ \cup V^0 \cup G$ <u>and</u> $V^- \cup V^0 \cup G$ <u>, respectively, with pairwise intersections of lower dimensions</u>.

It is easy to see that $\Delta^-(C_1, \ldots, C_{n-1})$ is (n-2)-dimensional, for it is convex and contains the (n-2)-simplex $\Delta(C_1, \ldots, C_k, C_{k+1}, \ldots, C_\ell, C_{1,\ell+1}, \ldots, C_{1,n-1})$. The extreme points of $\Delta^-(C_1, \ldots, C_{n-1})$ are precisely the set $V^- \cup V^0 \cup G$. (Note that the elements of $V^- \cup V^0$, being extreme points from the start, are still extreme. The points of G must also be extreme since they lie on distinct edges of $\Delta(C_1, \ldots, C_{n-1})$, each having one of its endpoints not in $\Delta^-(C_1, \ldots, C_{n-1})$.)

Our proof of the existence of the partitioning described above is based on the special case where $V^0 = \phi$. But this assumption is not so restrictive, for it can be achieved by a small perturbation of C_n (which tilts H_0 slightly). Once the partition of the perturbed set is obtained, reversing the process causes some of the simplices in the partitioning to disappear; what remains is still a partitioning of the required type.

Assume $V^0 = \phi$. Then $\Delta^-(C_1, \ldots, C_{n-1})$ is a <u>frustum</u> of a simplex. Indeed the hyperplane H_0 separates the vertices $C_1, \ldots, C_{n-1}$ into two sets: V^- (of cardinality k) and V^+ (of cardinality $n-1-k$). Following Sommerville [11, p. 103], we refer to $\Delta^-(C_1, \ldots, C_{n-1})$ as a <u>frustum of type</u> $(k|n-1-k)$, the other frustum

$$\Delta^+(C_1, \ldots, C_{n-1}) := H_0^+ \cap \Delta(C_1, \ldots, C_{n-1})$$

being of type $(n-1-k|k)$. The first entry in the symbol $(\cdot|\cdot)$ represents the number of vertices of the <u>original</u> simplex in the frustum we

"keep" and the second entry is the number of vertices "cut off" by H_0. Sommerville shows that these two frusta are isomorphic to simplotopes, that is, Cartesian products of simplices. In particular, a frustum of type $(r|s)$ is isomorphic to $\Delta_{r-1} \times \Delta_s$. To illustrate, consider the frustum of type $(3|1)$ which arises when a plane section cuts one vertex off of a tetrahedron. The frustum is clearly isomorphic to a prism, i.e., $\Delta_2 \times \Delta_1$.

The objects we actually partition will be simplotopes, $\Delta_r \times \Delta_s$. We may assume that $r \leq s$ since the order of the factors is not essential. However, we do want to be sure that once this convention is extablished, then $\Delta_r \times \Delta_s$ and $\Delta_{r'} \times \Delta_{s'}$ are isomorphic if and only if $r = r'$ and $s = s'$. Now if $\Delta_r \times \Delta_s$ and $\Delta_{r'} \times \Delta_{s'}$ are isomorphic, they have the same number of vertices and faces (of maximal dimension). The number of vertices is

$$(r + 1)(s + 1) = (r' + 1)(s' + 1) .$$

Thus

$$rs + r + s = r's' + r' + s' .$$

The number of faces is

$$r + s + 2 = r' + s' + 2 .$$

Therefore, $\Delta_r \times \Delta_s$ is isomorphic to $\Delta_{r'} \times \Delta_{s'}$ if and only if

$$rs = r's'$$
$$r + s = r' + s'$$
$$r \leq s$$
$$r' \leq s'$$

These conditions hold if and only if $r = r'$ and $s = s'$. See Figure 7.

<u>Theorem</u> 3. The simplotope $\Delta_r \times \Delta_s$ can be partitioned into $\binom{r+s}{r} = \binom{r+s}{s}$ simplices of dimension $r+s$.

<u>Proof</u>. The $r + s + 2$ faces of $\Delta_r \times \Delta_s$ are of two types. Those of the first type (of which there are $r + 1$) are of the form $\Delta_{r-1} \times \Delta_s$. Those of the second type (of which there are $s + 1$) are of the form $\Delta_r \times \Delta_{s-1}$. The generation of $\Delta_r \times \Delta_s$ can be performed as follows. (See Sommerville [11, p. 113].) Let P_0 denote a vertex of Δ_r. Each $(r-1)$-face of Δ_r generates an $(r+s-1)$-face of $\Delta_r \times \Delta_s$ when P_0 is moved all over Δ_s. This gives rise to $r + 1$ $(r+s-1)$-faces of the form $\Delta_{r-1} \times \Delta_s$. Also, Δ_r generates an $(r+s-1)$-face of $\Delta_r \times \Delta_s$ when P_0 is moved all over a $(s-1)$-face of Δ_s. This gives rise to $s + 1$ $(r+s-1)$-faces of the form $\Delta_r \times \Delta_{s-1}$. It is clear from the way the product is generated that the intersection of an $(r+s-1)$-face of the first type and an $(r+s-1)$-face of the second type is an $(r+s-2)$-face of type $\Delta_{r-1} \times \Delta_{s-1}$. Furthermore, the intersection of two $(r-1)$-faces of Δ_r is a $(r-2)$-face of the form Δ_{r-2}. Hence two $(r+s-1)$-faces of the first (second) type have as intersection an $(r+s-2)$-face of the form $\Delta_{r-2} \times \Delta_s$ $(\Delta_r \times \Delta_{s-2})$.

Now let F_1 be an $(r+s-1)$-face of the first type and let F_2 be a $(r+s-1)$-face of the second type. Then there exists exactly one vertex U of $\Delta_r \times \Delta_s$ which does not lie on $F_1 \cup F_2$. By the above, $F_1 \cap F_2$ is a $(r+s-2)$-face F_3 of type $\Delta_{r-1} \times \Delta_{s-1}$. Now

$$\text{Face } \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \text{ has } \begin{Bmatrix} r(s+1) \\ (r+1)s \\ rs \end{Bmatrix} \text{ vertices .}$$

Hence $F_1 \cup F_2$ has $r(s+1) + (r+1)s - rs = (r+1)(s+1) - 1$ vertices, exactly one less than $\Delta_r \times \Delta_s$. Using U as apex, form the pyramids $\hat{F}_1$ and $\hat{F}_2$ on F_1 and F_2 , respectively. These have a common (r+s-1)-face $\hat{F}_3$, the pyramid over $F_3 := F_1 \cap F_2$. Now the hyperplane generated by $\hat{F}_3$ cuts $\Delta_r \times \Delta_s$ into two pieces K_1 and K_2 . It follows readily from the definitions of $\hat{F}_1$, $\hat{F}_2$, $\hat{F}_3$ and the convexity of these pyramids and of K_1 , K_2 and $\Delta_r \times \Delta_s$ that

$$K_1 = \hat{F}_1 , \quad K_2 = \hat{F}_2 , \text{ and } K_1 \cap K_2 = \hat{F}_1 \cap \hat{F}_2 .$$

Hence

$$\hat{F}_1 \cup \hat{F}_2 = \Delta_r \times \Delta_s .$$

The partitioning process is now clear. Given $\Delta_r \times \Delta_s$ there are faces F_1 , F_2 of the two types and a unique vertex V contained in neither one and corresponding pyramids $\hat{F}_1$, $\hat{F}_2$ such that $\hat{F}_1 \cup \hat{F}_2 = \Delta_r \times \Delta_s$. Taking partitions of F_1 and F_2 in the obvious inductive manner induces simplicial partitionings of $\hat{F}_1$ and $\hat{F}_2$. Assembling these we get the required partitioning of $\Delta_r \times \Delta_s$. (See Figure 8.)

Finally, let $N(r, s)$ be the number of $(r+s)$-simplices this process produces in a partitioning of $\Delta_r \times \Delta_s$. Then clearly

$$N(r, s) = N(r-1, s) + N(r, s-1) = N(r, s) .$$

Since $\Delta_0 \times \Delta_s$ and $\Delta_r \times \Delta_0$ are already simplices,

$$N(0, s) = N(r, 0) = 1$$

and since the rectangle $\Delta_1 \times \Delta_1$ splits into two triangles (i.e., pyramids on two Δ_1's)

$$N(1, 1) = N(0, 1) + N(1, 0) = 2 .$$

Thus, the formula

$$N(r, s) = \binom{r+s}{r} = \binom{r+s}{s} \tag{32}$$

holds when $r + s$ equals 1 or 2. By induction and the identity

$$\binom{u}{v} = \binom{u-1}{v} + \binom{u-1}{v-1} ,$$

the formula follows for all nonnegative values of r and s. $\square$

Finally, we note that due to the "symmetry" of $\Delta_r \times \Delta_s$, the vertex U (referred to in the proof above) can be picked arbitrarily out of the set of $(r+1)(s+1)$ vertices of $\Delta_r \times \Delta_s$. Moreover, we can construct the partitioning of $\Delta_r \times \Delta_s$ in such a way that all simplices of the

triangulation have the same <u>two</u> vertices in common. Indeed, pick a vertex U and an $(r+s-1)$-face F_1 of the first type and an $(r+s-1)$-face F_2 of the second type such that $U \notin F_1 \cup F_2$. Pick a vertex V of the $(r+s-2)$-face $F_1 \cap F_2$, which is of type $\Delta_{r-1} \times \Delta_{s-1}$. Then we can use V as the apex of the pyramids in the inductive partitioning of F_1 and F_2. Hence all the simplices of the triangulations of $\hat{F}_1$ and $\hat{F}_2$ have V and U as vertices. (See Figure 8.) Note, however, that while U can be picked arbitrarily, the choice of V is not completely arbitrary, for after V has been chosen there are only rs possible candidates for V as U determines F_1 and F_2 uniquely. This can be made clear as follows.

Let the vertices of Δ_r and Δ_s be denoted $a_0, a_1, \ldots, a_r$ and $b_0, b_1, \ldots, b_s$, respectively. Then the vertices of $\Delta_r \times \Delta_s$ can be represented by the rectangular array $\{(a_i, b_j)\colon 0 \leq i \leq r\ ,\ 0 \leq j \leq s\}$:

$$\begin{array}{cccc} (a_0, b_0) & (a_0, b_1) & \cdots & (a_0, b_s) \\ (a_1, b_0) & (a_1, b_1) & \cdots & (a_1, b_s) \\ \vdots & \vdots & & \vdots \\ (a_r, b_0) & (a_r, b_1) & \cdots & (a_r, b_s) \end{array} \tag{33}$$

Without loss of generality, we may assume that $U = (a_0, b_0)$. Then clearly the vertices of F_1 are given by the subarray given by deleting the first row of the array (33). The vertices of F_2 are given by deleting the first column of (33). Thus the vertices of $F_1 \cap F_2$ are given by the subarray obtained by deleting the first row and column of (33). This being done, V may be chosen as (a_r, b_s).

Now, following a suggestion privately communicated to us by M. Hazewinkel, we may conveniently describe a procedure for triangulating

$\Delta_r \times \Delta_s$ which at the same time gives a "geometrical" explanation for the formula (32). Consider the set of all paths in the array (33) from the upper left-hand corner to the lower right-hand corner which move right or down only. For each such path, the indices of the corresponding array entries form a lexicographically increasing sequence of length $r + s + 1$, and the corresponding vertices of $\Delta_r \times \Delta_s$ determine a $(r+s)$-simplex. The number of distinct paths of this kind is readily seen to be $\binom{r+s}{r}$, and the corresponding $(r+s)$-simplices all have the vertices $U = (a_0, b_0)$ and $V = (a_r, b_s)$ in common. They in fact yield a partitioning of $\Delta_r \times \Delta_s$ as described above, as we shall now show. Any two of the $(r+s)$-simplices corresponding to two distinct paths of the above kind can have at most $r + s$ vertices in common as the corresponding paths can have at most $r + s$ entries of the array in common. Hence all the simplices corresponding to the above paths have pairwise disjoint interiors, i.e., have at most a common $(r+s-1)$-face. To show that they in fact exhaust $\Delta_r \times \Delta_s$ we proceed as follows. Clearly their union is contained in $\Delta_r \times \Delta_s$. Now choose an arbitrary point $(u, v) \in \Delta_r \times \Delta_s$. Its membership in one of the $(r+s)$-simplicies can be determined by applying the so-called Northwest Corner Rule to the transportation problem [3, p. 361] in which the barycentric coordinates of $u \in \Delta_r$ and $v \in \Delta_s$ are the "supplies" and "demands", respectively. The cells (i.e., array locations) of the basic variables so chosen correspond to the vertices of a $(r+s)$-simplex (in the triangulation of $\Delta_r \times \Delta_s$) which contains (u, v). Thus, we have a triangulation of $\Delta_r \times \Delta_s$. This is just the partitioning of $\Delta_r \times \Delta_s$ described earlier can be seen as follows. All the paths specified above which leave U vertically (horizontally) yield to a triangulation of F_1

(F_2) which is just the inductively-assumed partitioning of F_1 (F_2) described earlier. This provides a convenient scheme for determining a triangulation of $\Delta_r \times \Delta_s$.

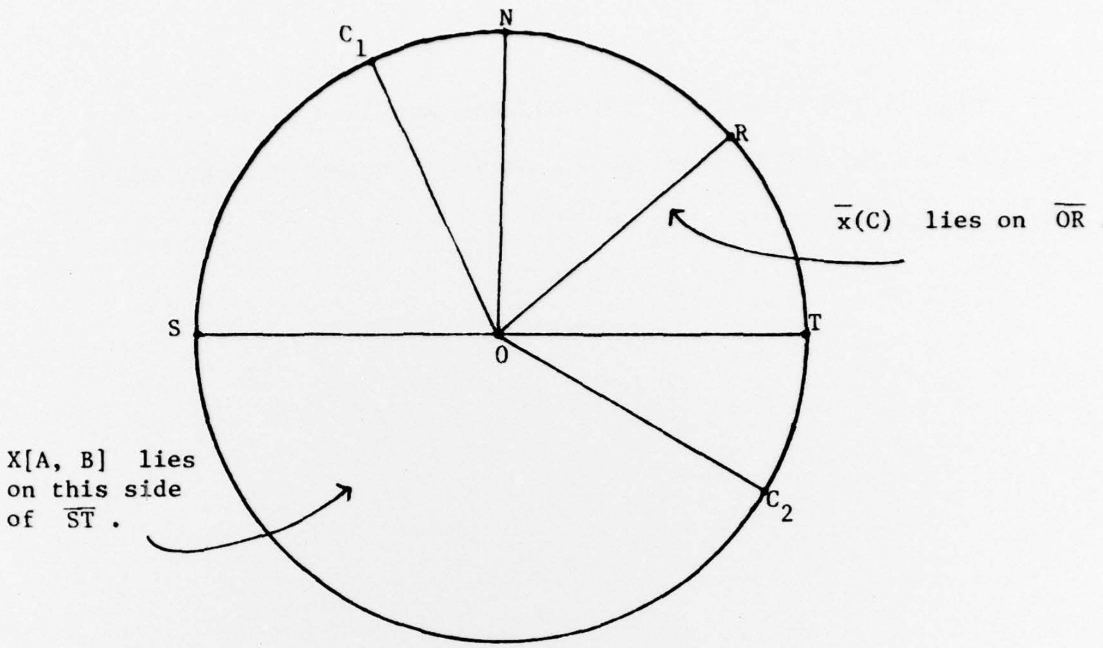

Figure 1

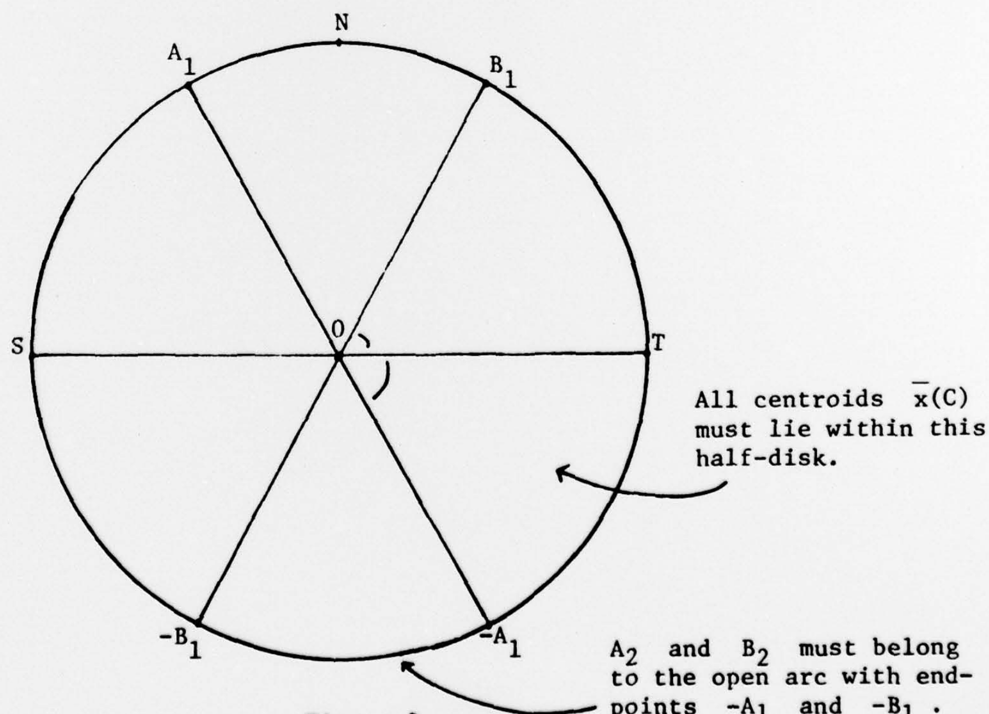

Figure 2

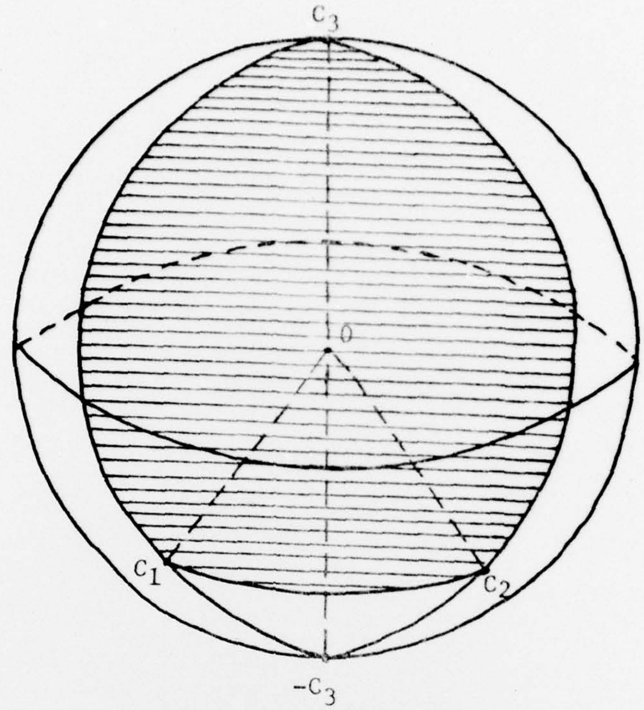

Figure 3

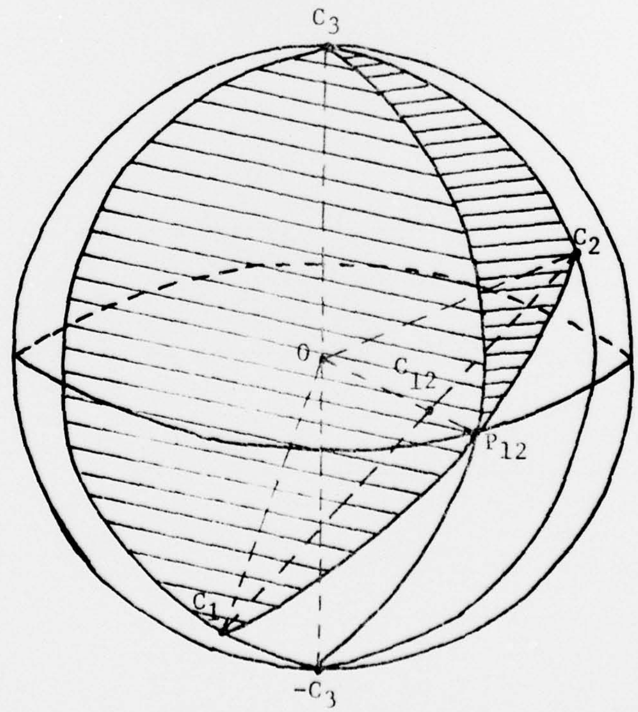

Figure 4

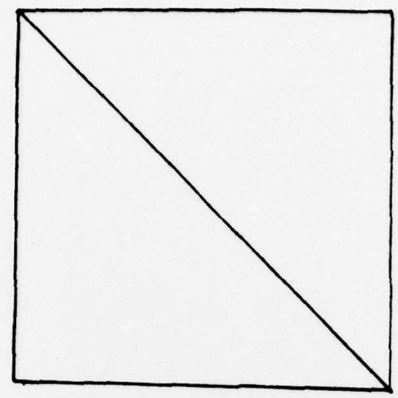

Figure 5

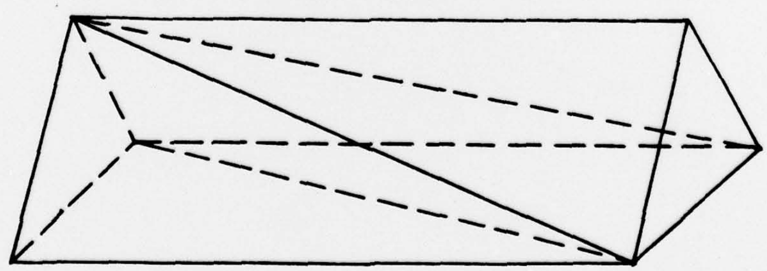

Figure 6

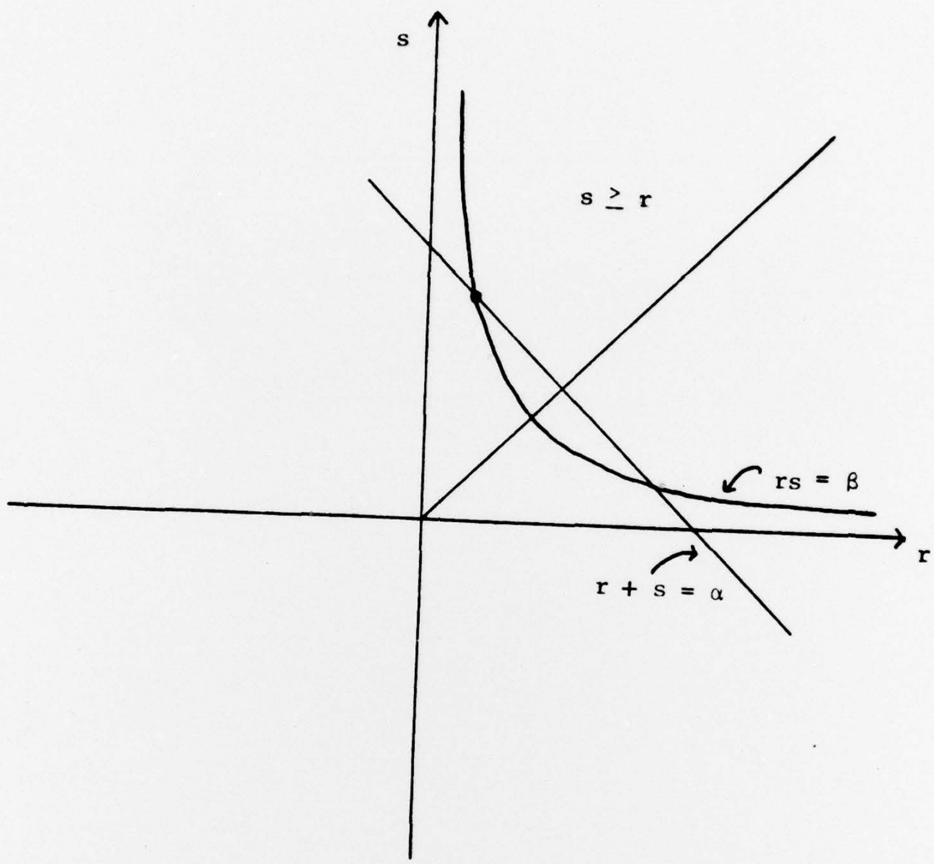

Figure 7

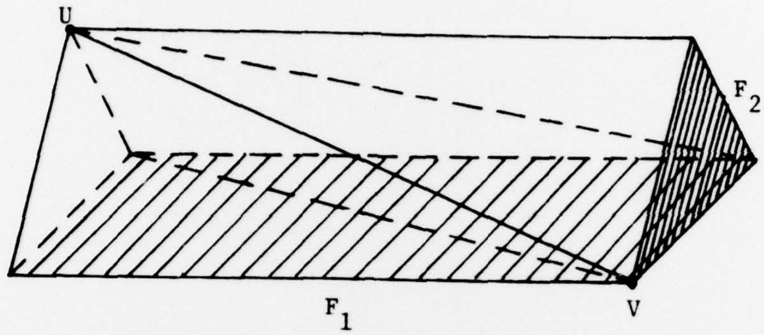

Figure 8

REFERENCES

1. Aganagic, M. and R. W. Cottle: "On Q-matrices," Technical Report SOL 78-9, Department of Operations Research, Stanford University, September 1978.

2. Bonnesen, T. and W. Fenchel: Theorie der konvexen Körper, Chelsea Publishing Company, New York, 1948.

3. Dantzig, G. B.: "Application of the simplex method to a transportation problem" in (T. C. Koopmans, ed.) Activity Analysis of Production and Allocation, John Wiley & Sons, New York, 1951, pp. 359-373.

4. Grünbaum, B.: Convex polytopes, Interscience Publishers, New York, 1967.

5. Kelly, L. M. and L. T. Watson: "Erratum: some perturbation theorems for Q-matrices," SIAM Journal of Applied Mathematics 34 (1978), pp. 320-321.

6. Kelly, L. M. and L. T. Watson: "Q-matrices and spherical geometry," Linear Algebra and Its Applications 25 (1979), 175-189.

7. Mangasarian, O. L.: "Characterization of linear complementarity problems as linear programs," Mathematical Programming Study 7, North-Holland Publishing Company, Amsterdam, 1978, pp. 74-87.

8. Murty, K. G.: "On the number of solutions to the complementarity problem and spanning properties of complementary cones," Linear Algebra and Its Applications 5 (1972), pp. 65-108.

9. Pang, J.-S.: "A note on an open problem in linear complementarity," Mathematical Programming 13 (1977), pp. 360-363.

10. Samelson, H., R. M. Thrall and O. Wesler, "A partition theorem for Euclidean n-space," Proceedings of the American Mathematical Society 9 (1958), pp. 805-807.

11. Sommerville, D. M. Y.: An introduction to the geometry of n dimensions, Dover Publications, New York, 1958.

12. Watson, L. T.: A variational approach to the linear complementarity problem, Ph.D. Thesis, University of Michigan, 1974.

13. Watson, L. T.: "Some perturbation theorems for Q-matrices," SIAM Journal of Applied Mathematics 31 (1976), pp. 379-384.

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79-10, Richard W. Cottle and Rabe von Randow
ON Q-MATRICES, CENTROIDS AND SIMPLOTOPES

S superscript (n-1) R superscript n

This paper establishes a necessary condition for a set of spherical (n-1)-simplicies to cover the sphere S^{n-1} in R^n. It is shown that the condition is also sufficient when $n = 2$ but is not so when $n > 2$. The result can be viewed as a property of Q-matrices, which arise in connection with the linear complementarity problem. It follows from two others also proved here. One is a partitioning theorem for a particular type of convex body known as a simplotope (the cartesian product of two simplices). The other says that the centroid of a suitable defined spherical sector has a positive inner product with each nonzero element of the sector.